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ON μ-RECURRENT NONSINGULAR ENDOMORPHISMS†

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ABSTRACT
We show that the Maharam skew product of μ-recurrent nonsingular endomorphisms is conservative and give some applications. Among them is the construction of a conservative ergodic invertible natural extension for μ-recurrent ergodic nonsingular endomorphisms.

1. Introduction

In this paper we are concerned with noninvertible nonsingular transformations acting on σ-finite measure spaces. Our first result (Theorem 2) is a generalization, to noninvertible transformations, of a theorem of Maharam [9] on the conservativity of a skew product. A direct consequence of this theorem is an extension of a result of Krieger [8] (see also Schmidt [15], [16]) on the recurrence of the Radon–Nikodym derivatives of conservative automorphisms. We then use Theorem 2 to construct a conservative ergodic natural extension for μ-recurrent ergodic nonsingular endomorphisms. (A different invertible extension of T, defined on $X \times [0, 1]$, was shown to the author by D. Maharam [10] in 1983. However, the proofs are entirely different and the extensions have different properties.)

A transformation $T: (X, \mu) \to (Y, \nu)$ is *nonsingular* provided $\mu(T^{-1}A) = 0$ if and only if $\nu A = 0$. A transformation T of a σ-finite measure space $(X, \mathscr{B}, \mu)$ is a *nonsingular automorphism* if it is invertible and T and T^{-1} are measurable and nonsingular. For any integer n, μT^n is a measure and there exist Radon–Nikodym derivatives $\omega_n(x) = d\mu T^n / d\mu(x)$. We usually write ω_1 as ω. One can show that the following relation holds a.e.:

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$$\omega_{i+j}(x) = \omega_j(x)\omega_i(T^j x). \tag{1}$$

A *nonsingular endomorphism* is a nonsingular transformation of X onto itself such that $T^{-1}A$ is measurable whenever A is measurable.

A nonnull set W is said to be *wandering* if $T^{-n}W \cap W = \varnothing$ for $n > 0$. A nonsingular endomorphism is *conservative* (or *incompressible*) if it admits no wandering sets; it is *measure preserving* if $\mu T^{-1}A = \mu A$ for all measurable sets A, and it is *ergodic* if whenever A is *T-invariant*, i.e. $T^{-1}A = A$, then $A = \varnothing$ (mod 0) or $A = X$ (mod 0). We note that in the noninvertible case one can have ergodic transformations that are not conservative. In fact, let S be defined on $\mathbf{N}$ by $S(n) = n - 1$, $S(1) = 3$, and let T be any irrational rotation of the unit interval I, then $S \times T$ acts ergodically on the product space with (non-atomic) product measure but is not conservative.

The definition of nonsingular endomorphism does not require TA to be measurable. However, in most applications one deals with (the completion of) standard Borel spaces, and in this case the measurability of TA follows if one assumes $\mu N = 0$ implies $\mu TN = 0$. In fact, if $A \in \mathscr{B}$ then $A = B \triangle N$ where B is Borel and N is null. Then $TA = T(B - N) \cup T(N - B)$; since TB is analytic it is (completion) measurable and since $T(B - N)$ differs from TB by a null set it is measurable. Nonsingularity gives that TA must be measurable.

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2. The skew product

We recall the following skew product first introduced in [9]. Let T be a nonsingular automorphism of (X, μ). Define

$$X^* = X \times \mathbf{R}^+, \quad T^*(x, y) = (Tx, y/\omega(x)), \quad \text{and} \quad \mu^* = \mu \times \lambda,$$

where λ is Lebesgue measure on $\mathbf{R}^+$. One can easily show that T^* is measure preserving and that if T^* is conservative, then so is T. In [9], Maharam proved the following theorem.

THEOREM 1. [9] *Let T be a nonsingular automorphism of a σ-finite measure space $(X, \mathscr{B}, \mu)$. T is conservative if and only if T^* is conservative.*

When T is an endomorphism, μT need not be additive, but if we let $\mathscr{A} = T^{-1}\mathscr{B}$, then μT is a σ-finite measure when restricted to $\mathscr{A}$, and is equivalent to μ. However, to apply the Radon–Nikodym theorem it is necessary to assume that μ is σ-finite when restricted to the sub-σ-algebra $\mathscr{A}$. Henceforth in this paper, a nonsingular endomorphism is assumed to satisfy this property (this is trivially true when μ is finite). In this case we define $\omega(x) = d\mu T/d\mu(x)$ as an $\mathscr{A}$-measurable function (cf. [3]). One defines the higher derivatives by

$$\omega_i(x) = \omega(x)\omega(Tx)\cdots\omega(T^{i-1}x) \qquad (i > 1).$$

Define now $T^*(x, y) = (Tx, y/\omega(x))$, and let the measure on the product space be $\mu \times \lambda$. When T is invertible, T^* is the original transformation.

There is an alternative approach to this definition. Write $\varphi(x) = d\mu T^{-1}/d\mu(x)$, $\theta(x) = \varphi(Tx)$ and define $T^*(x, y) = (Tx, y\theta(x))$. In order that φ be finite a.e. it is necessary to assume that μT^{-1} is σ-finite on $\mathscr{B}$ (this is the same as assuming that μ is σ-finite on $\mathscr{A}$). In general, write

$$\theta_1(x) = \theta(x), \quad \theta_{i+1}(x) = \theta(x)\theta_i(Tx) \qquad \text{for } i \geqq 1.$$

Since $\theta(x)$ is $\mathscr{A}$-measurable, by uniqueness of the derivatives one can show that $\omega(x) = 1/\theta(x)$ a.e.

One, of course, has the following lemma.

LEMMA 1. *T^* is measure preserving on $(X \times \mathbf{R}^+, \mu \times \lambda)$.*

We need a definition before stating our main result. A nonsingular endomorphism T of (X, μ) is said to be *μ-recurrent* if for every nonnegative measurable function g, $\Sigma_{i \geqq 0}\, g(T^i x)\omega_i(x)$ takes only the values 0 and ∞ a.e. The following lemma is well-known but we outline a proof since it is important in the proof of Theorem 3.

LEMMA 2. *If there exists a positive function $f \in L^1(X, \mu)$ with $\Sigma_{i \geqq 0} f(T^i x)\omega_i(x) = \infty$ then T is μ-recurrent. Hence, in finite measure, μ-recurrence is equivalent to $\Sigma_{i \geqq 0}\, \omega_i(x) = \infty$.*

PROOF. Using the Hopf maximal lemma ([6], [7]) as in the proof of Lemma 8.4 in [6] one obtains that for every integrable $g \geqq 0$, $\Sigma_{i \geqq 0}\, g(T^i x)\omega_i(x) = 0$ or ∞. The result for measurable $g \geqq 0$ follows by approximation.

We observe that when T is invertible, it is well-known that if it is conservative then it is ν-recurrent for any measure ν equivalent to μ; this is not so in the noninvertible case, even in finite measure, as was shown by Tsurumi in [17]. Another example is given by the modified Boole transformation $Tx = \frac{1}{2}(x - x^{-1})$ on the real line. One can verify that T preserves the Cauchy distribution $2(\pi(1 + x^2))^{-1}dx$ and hence is conservative. (T is even exact [1], [11] and hence mixing of all degrees [14].) If $\omega(x)$ denotes the Radon-Nikodym derivative of T with respect to Lebesgue measure λ, by direct computation (or using the well-known fact that Boole's transformation $x \to x - x^{-1}$ preserves λ), one obtains that $\omega(x) = \frac{1}{2}$ and therefore that T is not λ-recurrent.

Tsurumi's example, and the modified Boole transformation are examples of conservative nonsingular endomorphisms in finite and σ-finite measure spaces, respectively, which are not λ-recurrent. However they both admit finite invariant measures. It would be of interest to find out whether ergodic conservative nonsingular endomorphisms not admiting σ-finite invariant measures are μ-recurrent.†

LEMMA 3. *A μ-recurrent nonsingular endomorphism is conservative.*

PROOF. Suppose there exists a wandering set W. From μ-recurrence it follows that the function $\Sigma_{i \geqq 0} \chi_W(T^i x)\omega_i(x)$ takes only the values 0 and ∞. But since W is wandering, when $x \in W$, $T^i x \notin W$ for $i > 0$, hence $\Sigma_{i \geqq 0} \chi_W(T^i x)\omega_i(x) = 1$, a contradiction.

THEOREM 2. *Let T be a nonsingular endomorphism of a σ-finite measure space. T is μ-recurrent if and only if T^* is conservative.*

Theorem 2 clearly obtains Theorem 1 in the case when T is invertible.

Before proceeding with the proof we need some definitions and a few technical lemmas. Firstly, let us say that a measurable set A is a *sweep-out set* for T if it has positive measure and $X = \bigcup_{i \geqq 0} T^{-i}A$ (mod 0). The proof of the following lemma is obtained by inducing on the set A.

LEMMA 4. *Let T be a measure preserving endomorphism of a σ-finite measure space (X, μ). If T admits a sweep-out set A of finite measure then T is conservative.*

† *Added in proof.* After submission of this paper, the author and Stanley Eigen have shown that any n-to-1 ($n > 1$) conservative ergodic endomorphism of a Lebesgue space admits an equivalent nonrecurrent measure.

LEMMA 5. *Let T be a μ-recurrent nonsingular endomorphism of a σ-finite measure space. If g is a nonnegative finite measurable function such that $g(x) \geqq g(Tx)\omega(x)$ a.e. then $g(x) = g(Tx)\omega(x)$ a.e.*

PROOF. Define P by $Pf(x) = f(Tx)\omega(x)$. It follows that $\Sigma_{k=0}^{N} P^k(g - Pg) = g - P^{N+1}g \leqq g$. Hence $\Sigma_{k=0}^{\infty} P^k(g - Pg) < \infty$. Then μ-recurrence implies that $g - Pg = 0$.

We note that conservativity of T alone is not sufficient in Lemma 5. In fact, if $g(x) = \Sigma_{k=0}^{\infty} \omega_i(x) < \infty$ one can verify that g satisfies $g(x) = g(Tx)\omega(x) + 1$.

We introduce some temporary notation to be used in the following lemma. Let ω_μ denote the Radon–Nikodym derivative of T with respect to the measure μ. We say that cocycles ω_μ and ω_ν are *cohomologous* if there exists a positive finite measurable function h such that $\omega_\mu h = \omega_\nu(h \circ T)$. As in the invertible case, it is clear that cohomologous cocycles give rise to isomorphic skew products (see e.g. [15], Lemma 5.1). (We note that our notation for cocycles, cohomology, etc. is multiplicative — otherwise it agrees with that of e.g. [15].) The following lemma is well-known for invertible transformations (see e.g. [5]).

LEMMA 6. *Let T be conservative and suppose there exists a positive finite function F satisfying $F(x) = F(Tx)\omega(x)$ a.e. Then T^* is conservative.*

PROOF. Define an equivalent measure ν by $\nu A = \int_A F d\mu$. Then ν is invariant and $\omega_\nu = 1$ a.e. Let $h = 1/F$. Then we have that $\omega_\mu h = \omega_\nu h \circ T$ and hence ω_μ and ω_ν are cohomologous ($\omega_\mu = \omega$). The skew product corresponding to μ is T^* and that corresponding to ν is $T^+(x, y) = (Tx, y)$ which is clearly conservative. The fact that T^* is isomorphic to T^+ completes the proof.

LEMMA 7. *Suppose T is conservative. Let $\mathcal{F}$ be the family of all invariant sets of positive measure Z for which there is a function F such that $0 < F(x) < \infty$ and $F(x) = F(Tx)\omega(x)$ for x in Z. If $\mathcal{F}$ is nonempty then it has a maximal element (under inclusion).*

PROOF. By disjointifying, any countable union of elements of $\mathcal{F}$ is in $\mathcal{F}$ (by conservativity, subinvariant sets are invariant). Since any (proper) chain of sets of positive measure is countable, it follows that any chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Zorn's Lemma implies that there is a maximal element in $\mathcal{F}$.

PROOF OF THEOREM 2. Suppose that T is μ-recurrent. We break up the proof into two parts. First assume that there is no T-invariant subset Z of positive measure for which there is a function F satisfying $0 < F(x) < \infty$ and $F(x) = F(Tx)\omega(x)$ for a.e. x in Z. We shall apply Lemma 4 to (X^*, μ^*, T^*). Let f be a positive function in $L^1(X)$ and write $A = \{(x, y): 0 < y < f(x)\}$. Clearly A has finite measure. We claim that A is a sweep-out set for T^*; then Lemma 4 will obtain the theorem. To prove the claim define $F(x) = \sup_{i \geqq 0}\{f(T^i x)\omega_i(x)\}$. One can verify that $F(x) \geqq F(Tx)\omega(x)$. Let $B = \{x : F(x) = \infty\}$ and $Z = X - B$; then $Z \subset T^{-1}Z$ and the conservativity of T (Lemma 3) implies $T^{-1}Z = Z$. Lemma 5 applied to T restricted to Z gives that $F(x) = F(Tx)\omega(x)$ on Z, which contradicts the assumption unless $\mu Z = 0$. Thus $B = X \pmod 0$ and A is a sweep-out set for T^*.

Now assume that there exists an invariant subset Z of positive measure with a positive finite function F satisfying

$$F(x) = F(Tx)\omega(x) \quad \text{for a.e. } x \text{ in } Z.$$

From Lemma 7, Z can be assumed maximal under inclusion. Then on $X - Z$, T satisfies the assumption of the first part of the proof and hence T^* is conservative on $(X - Z)^* = X^* - Z^*$ ($X - Z$ is subinvariant and hence invariant). It suffices to show that T^* is conservative on Z^*. Thus without loss of generality we can assume now that $Z = X$. Lemma 6 then applies and gives that T^* is conservative.

Now we show the converse. The conservativity of T^* implies that for any nonnegative measurable function f^* in X^*, $\Sigma_{i \geqq 0} f^*(T^{*i}(x, y))$ takes only the values 0 or ∞ a.e. Given a nonnegative measurable function f in X put $f^*(x, y) = (1/y)f(x)$. Since $f^*(T^{*i}(x, y)) = (1/y)f(T^i x)\omega_i(x)$ it follows that T must be μ-recurrent.

Remark 1. The proof can be simplified somewhat when T is assumed ergodic. In this case the first part of the proof works under the assumption that there is no positive finite function F with $F(x) = F(Tx)\omega(x)$; the second part remains the same but Lemma 7 and the argument connecting the two parts are not necessary. The idea of putting the two parts together using Lemma 7 is from [9].

It follows from Lemma 6 and Theorem 2 that if (T, μ) admits such a function F as above then T is μ-recurrent. However, the existence of an invariant measure ν equivalent to μ does not necessarily imply the existence of a function F as above since $d\mu/d\nu$ need not be $T^{-1}\mathscr{B}$ measurable. The question of the existence of recurrent and nonrecurrent measures will be studied further in a subsequent paper jointly with Stanley Eigen.

3. Recurrence

We start with an extension of a theorem of Krieger [8] (see also Schmidt [15], [16]) on the recurrence of the Radon–Nikodym derivatives of conservative nonsingular automorphisms. (The argument in the first part of the proof below also appears in [5] for the case of invertible transformations.) The (multiplicative) cocycle ω is said to be *recurrent* (cf. [16]) if for any $\varepsilon > 0$ and for almost all x there are infinitely many nonnegative integers i with $|\omega_i(x) - 1| < \varepsilon$.

THEOREM 3. *Let T be a nonsingular endomorphism of a σ-finite measure space. If T is μ-recurrent then ω is recurrent. When the measure of the space is finite, if ω is recurrent then T is μ-recurrent.*

PROOF. Suppose T is μ-recurrent. Given $\varepsilon > 0$ let $c = 1 + \varepsilon$ and $A^* = X \times [1, c]$. Then $\mu^* A^* > 0$. Since T^* is conservative (Theorem 2), Poincaré recurrence for conservative nonsingular endomorphisms (see, e.g. [7]) implies that for almost all (x, y) in A^* and for infinitely many integers i, $T^{*i}(x, y) \in A^*$. From equation (1) one can see that $T^{*i}(x, y) = (T^i x, y/\omega_i(x))$. Thus, for infinitely many integers i,

$$y/c < \omega_i(x) < y.$$

It follows that one can choose $y \in (1, c)$ almost arbitrarily so that for almost all x in X, $(x, y) \in A^*$. To complete the proof take $y \in (1 + \varepsilon/2, 1 + \varepsilon)$.

To show the converse assume $\mu X < \infty$. If ω is recurrent then the series $\Sigma_{i \geqq 0} \omega_i(x)$ must diverge to ∞ a.e. Lemma 2 now completes the proof.

When X is σ-finite, even if T is invertible, one does not have the second part of the theorem. In fact, let $Tx = x + 1$ be defined on the real line with Lebesgue measure, then $\omega(x) = 1$, but T is not conservative.

We obtain a Halmos–Ornstein Jacobian theorem for nonsingular endomorphisms as a direct consequence of Theorem 3. The result with the inequality in the opposite direction from the one in the corollary below is known and does not need the μ-recurrence assumption (see, e.g. [7], p. 20). However, for the other direction μ-recurrence is needed as can be seen by considering the modified Boole transformation mentioned earlier.

COROLLARY 1. *If T is a μ-recurrent nonsingular endomorphism of a σ-finite mesure space and if $\omega(x) \leqq 1$ a.e. then $\omega(x) = 1$ a.e.*

PROOF. Suppose $\omega(x) < 1$ on a set of positive measure. The conservativity T implies that $\lim \omega_n(x) = 0$ on a set of positive measure. But this contradicts Theorem 3.

It clearly follows from Theorem 3 that if $\lim \omega_i(x) = 0$ or $\lim \omega_i(x) = \infty$ a.e. then T is not μ-recurrent. However we have the following.

COROLLARY 2. *If T is μ-recurrent then $\omega^*(x) = \limsup \omega_i(x) > 0$. When the measure of X is finite, if $\omega^*(x) > 0$ then T is μ-recurrent.*

PROOF. One can easily see that $\omega^*(Tx)\omega(x) = \omega^*(x)$. It follows that $B = \{x : \omega^*(x) = 0\}$ is T-invariant. If $\mu B > 0$ then ω restricted to B would not be recurrent, and thus T not μ-recurrent. Now assume X has finite mesure and $\omega^*(x) > 0$. Then $\Sigma_{i \geqq 0} \omega_i(x) = \infty$ and thus T is μ-recurrent.

Finally we mention another application of the idea in the proof of Theorem 3. By applying the well-known Lemma 8 below to T^*, and by arguing as in the proof of Theorem 3 one obtains the following Theorem 4. (In the case when T is invertible this is a well-known property of type III_1 nonsingular automorphisms (cf. [5], [15]).)

LEMMA 8. *If the σ-finite measure space X is a topological space with a countable base such that every nonempty open set has positive measure, and if T is a conservative ergodic measure preserving endomorphism of X, then for almost every x in X the sequence $\{T^n x : n \geqq 0\}$ is dense.*

THEOREM 4. *Let $(X, \mathscr{B}, \mu)$ be a standard Borel space with a σ-finite measure μ on $\mathscr{B}$ giving positive measure to open sets. If T is a μ-recurrent nonsingular endomorphism of X such that T^* is ergodic, then for almost every x in X the sequence $\{\omega_i(x) : i \geqq 0\}$ is dense in $\mathbf{R}^+$.*

4. Natural extension

In this section we discuss an invertible extension for nonsingular endomorphisms and some of its dynamical properties. As remarked in the introduction, a different extension has been obtained by D. Maharam.

From now on we deal exclusively with standard Borel spaces $(X, \mathscr{B})$ (called separable standard Borel in [13]) with a (complete) σ-finite measure μ defined on them.

In [14], Rohlin constructed a natural (invertible) extension for finite measure preserving endomorphisms. We outline his construction here and point out that it also works in infinite measure.

Let T be a nonsingular endomorphism of $(X, \mathscr{B}, \mu)$. Define

$$X' = \{(x_i)_{i \geqq 0} : Tx_{i+1} = x_i,\ i \geqq 0\}$$

and $T' : X' \to X'$ by

$$T'(x_i) = (Tx_i) = (Tx_0, x_0, x_1, \ldots).$$

Given $A \in \mathscr{B}$ define cylinder sets in X' by $A^{(k)} = \{(x_i) \in X' : x_k \in A\}$. Let $\mathscr{A}'$ consist of all sets of the form $A^{(k)}$ for $A \in \mathscr{B}$ and $k \geqq 0$, and $\mathscr{A}$ consist of all sets of the form $A^{(0)}$ for $A \in \mathscr{B}$; both $\mathscr{A}$ and $\mathscr{A}'$ are algebras (note that for $i > k$, $A^{(k)} = (T^{-i+k}A)^{(i)}$). Let $\mathscr{B}'$ be the σ-algebra generated by $\mathscr{A}'$; it follows that $\mathscr{B}' = \bigvee_{i=0}^{\infty} T'^{i} \mathscr{A}$. A set in $\mathscr{B}'$ that happens to belong to $\mathscr{A}$ will be called *T-measurable*. One has that $(X', \mathscr{B}')$ is a standard Borel space ([14], Theorem V.2.5). (All this notation will be used throughout. We note that these definitions do not depend on any measure on X.) Assume now that T is a measure preserving endomorphism. Define a measure μ' on $\mathscr{A}'$ by $\mu' A^{(k)} = \mu A$. It can be shown that μ' is countably additive and hence has a unique extension to a σ-finite measure μ' on $\mathscr{B}'$. This is well-known when $\mu X < \infty$. Choksi in [2] has pointed out that inverse limits also exist for infinite σ-finite spaces, from which it follows that this construction also works when $\mu X = \infty$. Th exact formulation of the theorem that we need does not appear in [2] but can be deduced from Theorem 3.1 of [2] by standard methods (see, e.g. [4] Theorem 4.1, and the references cited in this paper), or from Theorem 2.2 of [2] by proceeding as in the proof of Theorem V.3.2 of [13]. However, we outline below a less technical proof.

We refer to [13] for definitions concerning inverse systems. By Theorem V.3.2 of [13], for any inverse system of standard Borel spaces with consistent finite measures $(Y_k, \mathscr{B}_k, \pi_k, \nu_k)$ (where $\pi_k : Y_k \to Y_{k-1}$) there exists a unique finite measure ν' on its inverse limit Y'. Now if $\mu X = \infty$ write $X' = \bigcup_{n \geqq 0} X_n^{(0)}$ where $X = \bigcup_{n \geqq 0} X_n$, with $\mu X_n < \infty$. For each fixed n consider the inverse system of consistent finite measures:

$$(X_n, \mu) \leftarrow (T^{-1}X_n, \mu) \leftarrow (T^{-2}X_n, \mu) \leftarrow \cdots,$$

i.e., the system $(Y_k, \mathscr{B}_k, \pi_k, \nu_k)$ where $Y_k = T^{-k}X_n$, $\mathscr{B}_k = \mathscr{B} \cap Y_k$, $\pi_k = T$, $\nu_k = \mu$, for all k and fixed n. By the result just mentioned there exist finite measures μ_n' on $(X_n^{(0)}, \mathscr{B}' \cap X_n^{(0)})$. Now define $\mu' A = \Sigma_{n \geqq 0} \mu_n'(A \cap X_n^{(0)})$ for

$A \in \mathscr{B}'$. This indeed gives a σ-finite measure μ' on $(X', \mathscr{B}')$ such that $\mu' A^{(0)} = \mu A$, and thus T' is a measure preserving automorphism on $(X', \mathscr{B}', \mu')$.

Now we construct our invertible extension. Start with a nonsingular endomorphism T on a σ-finite standard Borel space $(X, \mathscr{B}, \mu)$. Define T^*, X^*, μ^* as before. Now apply the Rohlin extension to this space to obtain a measure preserving automorphism $T^{*\prime}$ on $(X^{*\prime}, \mu^{*\prime})$. (Note that even though X^* is infinite σ-finite the $'$ construction applies since T^* is measure preserving.) We note that $(X', \mathscr{B}')$ and T' are well-defined as before. Now we define a measure on $\mathscr{B}'$. For $E \in \mathscr{B}'$ let

$$\mu' E = \sum_{n \geqq 1} 2^{-n} \mu^{*\prime}(\{(x_i, y_i): (x_i) \in E \;\&\; n - 1 < y_0 \leqq n\}).$$

We give an equivalent description of μ' that will prove useful later. First define the projections $\theta : X' \to X$ and $\varphi : X^{*\prime} \to X'$ by $\theta((x_i)) = x_0$ and $\varphi((x_i, y_i)) = (x_i)$. Define a measure ν equivalent to $\mu^{*\prime}$ by

$$\nu A = \sum_{n \geqq 1} 2^{-n} \mu^{*\prime}(A \cap \{(x_i, y_i) \in X^{*\prime} : n - 1 < y_0 \leqq n\}).$$

Then clearly $\mu' E = \nu \varphi^{-1} E$ for all $E \in \mathscr{B}'$.

Lemma 9. *The following diagram is commutative and for every $A \in \mathscr{B}$, $\mu A = \mu' \theta^{-1} A$. In particular, $\mu' X' = \mu X$.*

$$\begin{array}{c} T' : (X', \mu') \to (X', \mu') \\ \theta \downarrow \qquad\qquad \downarrow \theta \\ T : (X, \mu) \to (X, \mu) \end{array}$$

Proof. The commutativity is immediate. Let $A \in \mathscr{B}$. Then

$$\theta^{-1} A = \{(x_i) \in X' : x_0 \in A\} = A^{(0)}.$$

Now

$$\begin{aligned} \mu' A^{(0)} &= \sum_{n \geqq 1} 2^{-n} \mu^{*\prime}(\{(x_i, y_i) : (x_i) \in A^{(0)} \;\&\; n - 1 < y_0 \leqq n\}) \\ &= \sum_{n \geqq 1} 2^{-n} \mu^{*\prime}(\{(x_i, y_i) : x_0 \in A \;\&\; n - 1 < y_0 \leqq n\}) \\ &= \sum_{n \geqq 1} 2^{-n} \mu^{*\prime}(\{(x_i, y_i) : (x_0, y_0) \in A \times (n - 1, n]\}) \\ &= \sum_{n \geqq 1} 2^{-n} \mu^{*\prime}(\{A \times (n - 1, n]\}^{(0)}) \end{aligned}$$

$$= \sum_{n \geq 1} 2^{-n} \mu^*(A \times (n-1, n])$$

$$= \mu A.$$

COROLLARY 3. *For every* $A \in \mathscr{B}$,

$$\int_{\theta^{-1}A} f \circ \theta d\mu' = \int_A f d\mu.$$

Hence every $f \in L^1(X)$ *lifts to* $f \circ \theta \in L^1(X')$.

REMARK 2. We have given two definitions for the measure μ'. When (X, μ, T) is a measure preserving system then μ' denotes the Rohlin extension obtained by taking inverse limits. In the nonsingular case μ' denotes the measure given above. It follows from Lemma 9 that when the nonsingular endomorphism T happens to be measure preserving the second definition gives the same measure as the Rohlin definition.

LEMMA 10. [12] *If T is a conservative measure preserving endomorphism of* $(X, \mathscr{B}, \mu)$ *then T' is conservative.*

PROOF. Let $f > 0$ be an integrable function in X and put $f' = f \circ \theta$. Then $f' > 0$ and integrable, and if D is the dissipative part of T', since T' is a measure preserving automorphism the Hopf decomposition theorem gives that

$$D = \left\{ (x_i) \in X' : \sum_{k=0}^{\infty} f'(T'^k(x_i)) < \infty \right\}.$$

But $D = D_0^{(0)}$ where

$$D_0 = \left\{ x_0 \in X : \sum_{k=0}^{\infty} f(T^k x_0) < \infty \right\}.$$

Since T is conservative $D_0 = \varnothing$ (mod 0).

LEMMA 11. *If the nonsingular endomorphism T is μ-recurrent then T' is a conservative nonsingular automorphism on* $(X', \mathscr{B}', \mu')$.

PROOF. Let ν be the measure on $X^{*\prime}$ equivalent to $\mu^{*\prime}$ defined above. By Lemma 10, $T^{*\prime}$ is conservative nonsingular on $(X^{*\prime}, \mu^{*\prime})$, hence on $(X^{*\prime}, \nu)$. Then map $\varphi : (X^{*\prime}, \nu) \to (X', \mu')$ is a nonsingular measurable surjection such that $\varphi T^{*\prime} = T' \varphi$. Whence T' is conservative nonsingular on $(X', \mathscr{B}', \mu')$.

The proof of the following lemma is essentially from [12].

LEMMA 12. *Let T be a conservative measure preserving endomorphism. If E is T'-invariant then E is T-measurable, i.e., $E \in \mathcal{A}$.*

PROOF. We recall again that $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{B}'$ and $\bigvee_{i=0}^{\infty} T'^{i}\mathcal{A} = \mathcal{B}'$, $T'^{-1}\mathcal{A} \subset \mathcal{A}$, and T' is conservative.

Let $G \in \mathcal{B}'$ with $0 < \mu'G < \infty$ and $E_0 = E \cap G$. Then $T_G'^{-1}E_0 = E_0$, where T'_G is the induced transformation on G.

Since $\bigvee_0^{\infty} T_G'^{n}(\mathcal{A} \cap G) = \mathcal{B}' \cap G$, given $\varepsilon > 0$, there exists $F \in T_G'^{n}(\mathcal{A} \cap G)$, for some $n \geqq 0$, such that $\mu'(E_0 \triangle F) < \varepsilon$. Therefore $\mu'(E_0 \triangle T_G'^{-n}F) < \varepsilon$ where $T_G'^{-n}F \in \mathcal{A} \cap G$. Thus $E_0 \in \mathcal{A} \cap G$ (mod 0). By taking a disjoint sequence G_i of sets in $\mathcal{B}'$ of finite measure such that $X' = \bigcup G_i$ we get that $E \in \mathcal{A}$ (mod 0).

THEOREM 5. *Let T be a nonsingular endomorphism of a σ-finite standard Borel space $(X, \mathcal{B}, \mu)$. If T is μ-recurrent ergodic then it has a conservative ergodic invertible natural extension T' on $(X', \mathcal{B}', \mu')$.*

PROOF. It has already been shown that T' is a conservative nonsingular automorphism. Consider the following commutative diagram.

$$\begin{array}{ccccccc} X^{*\prime} & \xrightarrow{T^{*\prime}} & X^{*\prime} & \xrightarrow{\theta^*} & X^* & \xrightarrow{T^*} & X^* \\ \varphi\downarrow & & \varphi\downarrow & & \downarrow\pi & & \downarrow\pi \\ X' & \xrightarrow{T'} & X' & \xrightarrow{\theta} & X & \xrightarrow{T} & X \end{array}$$

where $\theta^*((x_i, y_i)) = (x_0, y_0)$, $\pi(x, y) = x$, and φ and θ are as defined earlier. It is clear that all the maps commute and φ, θ, π, θ^* are nonsingular surjections with respect to the usual measures (the measure on X' is $\mu' = \nu\varphi^{-1}$). Suppose now that A is T'-invariant. Then $\varphi^{-1}A$ is $T^{*\prime}$-invariant and by Lemma 12, $\varphi^{-1}A = B^{(0)} = \theta^{*-1}B$ for some $B \in \mathcal{B}^*$. Since $B = \theta^*(\varphi^{-1}A)$ then $B = \pi^{-1}B_0$ for some some $B_0 \in \mathcal{B}$. Now

$$A = \varphi \circ \theta^{*-1}B = \varphi \circ \theta^{*-1}\pi^{-1}B_0 = \theta^{-1}B_0 = B_0^{(0)},$$

and B_0 must be T-invariant. Since T is ergodic $B_0 = \emptyset$ or $B_0 = X$ (mod 0), which completes the proof.

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